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Влияние фонового течения в моделях длинных нелинейных внутренних волн

Уравнение Кортевега—де Вриза является стандартной моделью для описания динамики длинных нелинейных внутренних волн в океане. Когда принимаются во внимание слабые воздействия вращения Земли и поперечных возмущений, то уравнение изменяется к форме так называемого модифицированного уравнения Кадомцева—Петвиашвили с вращением. В этой короткой статье дана ревизия асимптотической процедуры вывода этого уравнения с учетом фонового сдвигового течения так же, как и фоновой стратификации.

Ключевые слова: солитоны внутренних волн, уравнение Кадомцева—Петвиашвили, уравнение Островского.

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Effect of a Background Shear Current on Models for Nonlinear Long Internal Waves

The Korteweg—de Vries equation is a standard model for the description of long nonlinear internal waves in the ocean. When the effect of weak transverse variations and the Earth's background rotation are taken into account, this is replaced by the rotation-modified Kadomtsev—Petviashvili equation. In this short note we revisit the asymptotic derivation of this equation, incorporating a background shear flow as well as the background stratification.

Key words: Internal solitary waves, Kadomtsev—Petviashvili equation, Ostrovsky equation.

It is well known that the internal solitary waves commonly observed in the coastal ocean can be modelled by the Korteweg—de Vries (KdV) equation, or a related equation, see the reviews by Grimshaw [1] and Helfrich and Melville [2] for instance. When expressed in a reference frame moving with the linear long wave speed c_0, the KdV equation is

$$A_t + \mu A A_x + \lambda A_{xxx} = 0.$$

Here x, t are space and time coordinates, and $A(x, t)$ is the amplitude of the linear long wave mode $\varphi(z)$ corresponding to the linear long wave phase speed c_0, which is determined from the modal equation, given by

$$(\rho_0 W_0^2 \varphi_z)z + \rho_0 N^2 \varphi = 0;\ W_0 = c_0 - u_0(z);\ -h < z < 0; \tag{1}$$

$$\varphi = 0 \text{ at } z = 0; \text{ and } W_0^2 \varphi_z = g\varphi \text{ at } z = 0. \tag{2}$$

Here $\rho_0(z)$, $u_0(z)$ are the background density and current respectively and $\rho_0 N^2 = -g\rho_{0z}$. The fluid occupies the domain between the rigid bottom $z = -h$ and the free surface at $z = 0$. The coefficients μ and λ are given by certain integrals involving the modal function,

$$I\,\mu = \int_{-h}^{0} \rho_0\, W_0^2 \varphi_z^3 dz\,,\ I\,\lambda = \int_{-h}^{0} \rho_0\, W_0^2 \varphi^2 dz\,,\ I = 2\int_{-h}^{0} \rho_0\, W_0\, \varphi_z^2\, dz\,.$$

However, oceanic internal waves are often observed to propagate for long distances over several inertial periods, and hence the effect of the Earth's background rotation needs to be taken into account. At the same time it may be necessary to take account of weak dependence on the transverse variable y. The simplest model equation which takes account of both these effects is the rotation-modified Kadomtsev—Petviashvili (KP) equation, see Grimshaw [3],

$$\{A_t + \mu A A_x + \lambda A_{xxx}\}_x + \beta A_{yy} - \gamma f^2 A = 0. \tag{3}$$

Here f is the Coriolis parameter measuring the Earth's rotation. This is an extension of the KP equation, see Kadomtsev and Petviashvili [4], which includes the y-variations on the one hand, and of the Ostrovsky equation, see Ostrovsky [5] for the original derivation or Grimshaw and Helfrich [6] for a recent account, which includes the rotational term on the other hand. In the absence of a background shear flow, that is $u_0(z) \equiv 0$,

$$\beta = \frac{c_0}{2}, \ \gamma = \frac{1}{2c_0}. \tag{4}$$

The explanation for these expressions lies in the linear dispersion relation for long waves, which is $\omega^2 \approx c_0^2(k^2 + l^2) + f^2$, for a frequency ω and wavenumbers k, l in the x, y-directions respectively. The dominant balance is $\omega \sim kc_0$, and moving to the reference frame moving with speed c_0 and adding the next order cubic linear dispersive correction, leads to the KdV equation. When weak y-dispersion ($l^2 << k^2$) and weak rotation are added, this becomes $\omega \sim kc_0 + c_0 l^2/2k + f^2/2kc_0$, and so leads to equation (3). A more formal asymptotic derivation was given by Grimshaw [3]. Note that here in the absence of a shear flow, $\lambda\beta > 0$, $\lambda\gamma > 0$ so that the KP part here is KPII and the rotational part is the regular Ostrovsky equation.

In this note we determine the coefficients β, γ when there is a background shear flow present. The expression for γ when there is no y-variation was derived by Grimshaw [7] using a formal asymptotic expansion, and here we reproduce that result using an alternative approach based on linear long wave theory. At the same time we derive the expression for β when there is a shear flow. But note that in the presence of background rotation a shear current leads to a term in the basic state, the Coriolis term $f\rho_0 u_0$ in the y-momentum equation, which in an inviscid conservative model needs to be balanced by a pressure gradient in the y-direction. However, this introduces a (weak) dependence on y, which is a complication we avoid here by supposing instead that this Coriolis term is balanced by a z-dependent body force. The case when there is a weak y-dependence will be addressed in a future study.

Rotational and transverse dispersion terms for a shear flow. Since both the terms of interest in equation (3) are linear and in the long wave regime, it is sufficient to consider only the linear long wave equations. Relative to a background shear flow u_0 and a background density field ρ_0, and on an f-plane, these are, in the domain $-h < z < 0$,

$$\rho_0(u_t + u_0 u_x + u_{0z} w - \varepsilon^2 F V + p_x = 0, \tag{5}$$

$$\rho_0(V_t + u_0 V_x + Fu) + F u_0 \rho + p_Y = 0, \tag{6}$$

$$p_z + g\rho = 0, \tag{7}$$

$$u_x + \varepsilon^2 VY + w_z = 0, \tag{8}$$

$$\zeta_t + u_0 \zeta_x - w = 0, \tag{9}$$

$$\rho_t + u_0 \rho_x + w\rho_{0z} = 0. \tag{10}$$

Here we have used a standard notation, ζ is the vertical particle displacement and we have put $Y = \varepsilon y$, $V = \varepsilon v$ and $F = \varepsilon f$ to represent the slow scales for transverse dependence and the background rotation. The boundary conditions are

$$w = 0; \text{ at } z = -h, \tag{11}$$

$$p - g\rho_0\zeta = 0\ ; \text{ at } z = 0. \tag{12}$$

It is convenient to look at the linear long wave theory in Fourier space, for a disturbance proportional to $\exp(ikx - ikct)$. Then equations (5)–(10) become, after eliminating w, ρ,

$$\rho_0(-ikW(u + u_{0z}\zeta) - \varepsilon^2 FV) + ikp = 0,$$

$$\rho_0(-ikWV + Fu) - F u_0 \rho_{0z}\zeta + pY = 0,$$

$$\rho_0 N^2 \zeta + pz = 0, \tag{13}$$

$$iku + \varepsilon^2 VY - ik(W\zeta)z = 0, \tag{14}$$

where $W = c - u_0$. Next we use (17) to eliminate u and so obtain in place of (14), (15),

$$\rho_0(-ikW^2\zeta_z + \varepsilon^2(WV_Y - FV)) + ikp = 0, \tag{15}$$

$$\rho_0 W(-ikV + F\zeta_z) - F(u_0\rho_0)_z\zeta + p_Y = 0. \tag{16}$$

Together with (13) these form three equations for ζ, p; V. The final step is to eliminate p between (13) and (15) to obtain

$$(\rho_0 W^2\zeta_z)_z + \rho_0 N^2\zeta = \varepsilon^2 G, -h < z < 0,$$

$$ikG = \{\rho_0(WV_Y - FV)\}_z.$$

The boundary conditions (11), (12) can likewise be reduced to

$$\zeta = 0 \text{ ; at } z = -h,$$

$$W^2\,\zeta_z - g\zeta = \varepsilon^2 H; \text{ at } z = 0,$$

$$ikH = WV_Y - FV.$$

Next, we expand in powers of ε^2,

$$\zeta = A\varphi(z) + \varepsilon^2\zeta_1 + \ldots,\ c = c_0 + \varepsilon^2 c_1 + \ldots,$$

Here we have anticipated that at the leading order we obtain the modal equations (1) and (2), and the amplitude A depends parametrically on k, c, Y. Then at the next order we get that

$$(\rho_0(W_0{}^2\zeta_{1z})_z + \rho_0 N^2\zeta_1 = G_1 = -2c_1(\rho W_0\varphi_z)_z + G, -h < z < 0, \tag{17}$$

$$\varphi_1 = 0, z = -h;\ W_0{}^2\varphi_{1z} - g\varphi_1 = H_1 = -2c_1 W_0\rho_0\varphi_z + H, z = 0. \tag{18}$$

Here the terms G, H are evaluated using the leading order expressions for ζ and p in (19), so that equations (17), (18) are a forced version of the modal equation. For a solution to exist, a compatibility condition must be satisfied. This is

$$\int_{-h}^{0} G_1\varphi dz = \left[\rho_0 H_1\varphi\right]_{z=0}. \tag{19}$$

Substituting from (17), (18) we get that

$$ikIc_1 A - \int_{-h}^{0} \rho_0\left(W_0 V_0 - FV\right)\varphi_z dz = 0, \tag{20}$$

while V can be evaluated to leading order from (16),

$$ik\rho_0 W_0 V = \rho_0 W_0{}^2\varphi_z A_Y - F(\rho_0 u_0)_z A\varphi + \rho_0 F W_0 A\varphi_z. \tag{21}$$

Substitution of (21) into (20) leads to

$$k^2 c_1 A + \beta A_{YY} + F\beta_1 A_Y - \gamma F^2 A = 0, \tag{22}$$

$$I\beta = \int_{-h}^{0} \rho_0 W_0^2\varphi_z^2 dz\ ,$$

$$I\beta_1 = -\int_{-h}^{0} \left(\rho_0 u_0\right)_z \varphi_z\varphi dz\,, \tag{23}$$

$$I\gamma = \int_{-h}^{0} \rho_0\Phi\,\varphi_z dz\ ,\ \rho_0 W_0\Phi = \rho_0 W_0\varphi_z - (\rho_0 u_0)_z\varphi.$$

We now convert the Fourier variables to physical space, using $kc_1 \leftrightarrow i\partial/\partial t$ and $k \leftrightarrow -i\partial/\partial$x. Restoring the nonlinear terms and the linear dispersive term we conclude that (22) is equivalent to

$$\{At + \mu AA_x + \lambda A_{xxx}\}_x + \beta A_{yy} + \beta_1 f A_y - \gamma f^2 A = 0. \tag{24}$$

Here we have restored the unscaled variables y; f.

Discussion. In this note we have extended the analysis of Grimshaw [7] for the Ostrovsky equation in the presence of a background shear flow to the rotation-modified KP equation. The outcome is equation (24). We see that in comparison to (3), there is an extra term, fA_y with coefficient β_1, due to the combined effect of rotation, transverse variations and the presence of a shear current.

In the absence of a shear current, $\beta_1 = 0$ and β, γ reduce to (4). The coefficient β of the term A_{yy} has the same sign as λ, and can be written in the form

$$\beta = \frac{c_0}{2} - \frac{1}{I}\int_{-h}^{0} u_0 W_0 \varphi_z^2 dz \,.$$

Hence the shear current provides a small correction term relative to $c_0/2$, which can be expected to be relatively positive or negative according as u0 is negative or positive. The amended coefficient γ of the term f^2A agrees with that obtained by Grimshaw [7] in the absence of y-variations. It can usually be expected to have the same sign as λ, but may have the opposite sign if the shear current is sufficiently strong, see Alias et al. [8].

The most interesting outcome here is the presence of the term in fA_y which arises only when there is rotation, a shear current and transverse variations. However, we caution that this extra term has this form because we have chosen here to balance the Coriolis term $f\rho_0 u_0$ in the basic state with a body force, rather than a pressure gradient in the y-direction. Nevertheless, its presence here implies that in the linearised theory the linear dispersion relation for solutions proportional to $\exp(ikx + ily - i\omega t)$ is

$$\omega k + \lambda k^4 - \beta l^2 + if\beta_1 l - \gamma f^2 = 0.$$

The complex term here implies that either there is an instability with growth rate $f\beta_1 l/k$, or that the y-wavenumber l is complex-valued with an imaginary part $i\beta_1 f/2\beta$. Neither consequence is satisfactory, implying that the more complicated theory is needed, when the Coriolis term $f\rho_0 u_0$ is balanced by a pressure gradient in the y-direction, with a consequent y-dependence of all terms in the basic state. But we note that in the latter case the offending term can be removed in the linearized equation by putting $A = A' \exp(-f\beta_1 y/2\beta)$, when the same equation is obtained, but there is no term in A_y' and the coefficient γ is changed to $\gamma - \beta_1^2/4\beta$. However, there is then a consequence that the nonlinear coefficient μ is replaced by $\mu \exp(-f\beta_1 y/2\beta)$.

Finally we note that the coefficient β_1 is non-zero even in the Boussinesq approximation, when ρ_0 in (23) can be regarded as constant, and at the upper boundary $\varphi \approx 0$. In this limit

$$\beta_1 \approx \int_{-h}^{0} \rho_0 u_{0zz} \frac{\varphi^2}{2} dz \,.$$

Thus β_1 is non-zero in general unless $u_{0zz} \equiv 0$.

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