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PARABOLIC TRAPS OF ROSSBY WAVES WAVES IN THE OCEAN

Received 24.04.2021; Revised 29.06.2021; Accepted 13.09.2021

Analysis of the Rossby wave dynamics shows when waves interact with shear currents vertical focusing of the modes occurs. Due to the inhomogeneity of the background flow, Rossby waves are captured by the current, and there is a compression of the modes on vertical horizons. For the vertical mode, instead of the classical trigonometric cosine, strongly localized solutions appear in the form of exponentially modulated Hermite polynomials. Qualitatively, the situation can be described as follows: an inhomogeneous background current acts like a kind of parabolic antenna. The wave, falling into this parabolic trap, begins to reflect off the narrowing walls of the paraboloid, while the vertical transparency zone narrows and the wave's progress towards the center of the paraboloid slows down more and more. In the linear formulation, this process lasts infinitely long, while the distance between adjacent reflection points from the paraboloid mirror gradually decreases. There is a mathematical description of this phenomenon for internal waves. Since there are no fundamental differences between internal waves and Rossby in the vicinity of the focus, the mathematical part of the work for internal waves can also be transformed for Rossby waves.

In this paper, in terms of the Fourier integral, we construct a two-dimensional analytical solution of the reference equation for the vertical focusing of a monochromatic wave in the vicinity of the focus. Using the degenerate hypergeometric function of the complex variable, we show the identity of this solution with the solution of the reference equation obtained in previous studies. We find the asymptotic behavior of the solution in the far zone by the stationary phase method. Using exponentially majored Hermite polynomials, we show the correct two-dimensional crosslinking of the obtained solution, which has in the form of a degenerate hypergeometric function of a complex variable, happens with the WKB solution in the far zone. We show the question of absorption in the focal zone is not unambiguous, and therefore both situations are possible: both the passage and the reflection from the feature.

Keywords: Rossby waves, parabolic antenna, trap, WKB approximation, crosslinking, asymptotics, focal zone, transparency, shadow regions

УДК 551.465

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ПАРАБОЛИЧЕСКИЕ ЛОВУШКИ ВОЛН РОССБИ В ОКЕАНЕ

Статья поступила в редакцию 24.04.2021, после доработки 29.06.2021, принята в печать 13.09.2021

Анализ динамики волн Россби показывает, что при взаимодействии их со сдвиговыми течениями возможны режимы, когда из-за неоднородности фонового течения волны Россби захватываются течением, при этом происходит вертикальная фокусировка — сжатие моды на некотором вертикальном горизонте. Для вертикальной моды вместо классического тригонометрического косинуса появляются сильно локализованные решения в виде экспоненциально модулированных полиномов Эрмита. Качественно ситуацию можно описать следующим образом: неоднородное фоновое течение действует как некая параболическая антенна. Волна, попадая в эту параболическую ловушку, начинает отражаться от сужающихся стенок параболоида, при этом вертикальная зона прозрачности сужается, а продвижение волны к центру параболоида все более и более замедляется. В линейной

Ссылка для цитирования: *Гневышев В.Г.*, *Белоненко Т.В.* Параболические ловушки волн Россби в океане // Фундаментальная и прикладная гидрофизика. 2021. Т. 14, № 4. С. 14—24. doi: 10.7868/S207366732104002X

For citation: *Gnevyshev V.G.*, *Belonenko T.V.* Parabolic Traps of Rossby Waves in the Ocean. *Fundamentalnaya i Prikladnaya Gidrofizika*. 2021, 14, 4, 14—24. doi: 10.7868/S207366732104002X

постановке этот процесс длится бесконечно долго, при этом расстояние между соседними точками отражения от зеркала параболоида постепенно сокращается. Для внутренних волн такое уравнение в окрестности фокуса существует. Поскольку в окрестности фокуса нет принципиальных отличий внутренних волн от Россби, то математическую часть работы для внутренних волн можно трансформировать и для волн Россби.

В терминах интеграла Фурье построено двумерное аналитическое решение эталонного уравнения для вертикальной фокусировки монохроматической волны в окрестности фокуса. Показана идентичность этого решения с решением эталонного уравнения в терминах вырожденной гипергеометрической функции от комплексного переменного, полученного в предыдущих исследованиях. Методом стационарной фазы найдена асимптотика решения в дальней зоне. Показано, что корректная двумерная сшивка полученного решения в виде вырожденной гипергеометрической функции от комплексного переменного происходит с ВКБ-решением в дальней зоне в терминах экспоненциально мажорированных полиномов Эрмита. Показано, что вопрос о поглощении в фокальной зоне не носит однозначный характер, и поэтому возможны обе ситуации: как прохождение, так и отражение от особенности.

Ключевые слова: волны Россби, параболическая антенна, ловушка, ВКБ-приближение, сшивка, асимптотика, фокальная зона, области прозрачности и тени

1. Introduction

One of the methods for studying the dynamics of waves in the ocean is called the “Vertical modes — horizontal rays” method [1]. Since the horizontal scales of Rossby waves are from tens to hundreds of kilometers, this approximation works well in the open ocean. If the stratification is assumed constant and the topography and baroclinic background currents are not taken into account, then the vertical mode of Rossby waves is determined by one stratification and does not depend on the β-parameter. In this regard, the Rossby wave becomes similar to an ordinary internal wave, and its vertical mode is an ordinary trigonometric function with the classical quantization of the eigenvalues of the Sturm-Liouville problem. In this formulation, the determining factor is the horizontal heterogeneity of the large-scale flow. Horizontal flow changes are the leaders in the task. First, a geometric skeleton is drawn, and only then a vertical mode is built at each point in space. In this case, the vertical mode is a kind of follower, strictly following the horizontal propagation of the ray [2]. However, along with the similarity of the problem sets, there are both qualitative and quantitative differences for internal and Rossby waves. The first difference between Rossby waves and internal waves is that there are two qualitatively different scenarios for the evolution of the rays of Rossby waves, which is essentially a consequence of the presence of the β-parameter in the problem, both in the case of a zonal background flow [3] and non-zonal flow [4]. For the non-zonal case, a qualitatively new scenario appears, associated with such a phenomenon as “overshooting”, i. e., the diving with adhering of the Rossby wave under the critical layer [5]. The second and most significant difference between internal waves and Rossby waves is as follows. For internal waves, adding baroclinicity to the background flow does not qualitatively change the scenario of the evolution of the wave packet. The infinite countable spectrum of the Sturm-Liouville boundary value problem with a trigonometric set of eigenfunctions smoothly transforms into a new infinite countable spectrum, but already with eigenfunctions in the form of exponentially dominated Hermite polynomials. In this case, of course, new phenomena such as vertical focusing and “non-dispersive” focusing appear [6]. But for internal waves, the focal point remains a kind of “black hole”, while the rays are the “leaders”, and the vertical modes are “strong followers” with a certain secondary role as vertical focusing and “non-dispersive” focusing [6].

For Rossby waves, the situation is qualitatively different, while different authors, using different approaches, come to the following qualitative conclusion. Considering the baroclinicity of the background current has a very strong effect on the spectrum of the vertical problem. In [7, 8], situations were observed in numerical calculations, when, depending on the direction of the wave, only a few first modes exist, or even a vertical mode may not exist at all.

The authors of [9], within the framework of a two-layer model, have convincingly proved that long baroclinic Rossby waves are unstable. This is shown on the basis of a laboratory experiments and a generalization of these results for the case of various media is also given in [10]. It is important to note that this again results in the stratification of the results into two cases: zonal and non-zonal. If for a zonal flow there are some upper bounds on the growth rates of instability [11, 12], then for the non-zonal case, due to the non-Hermitian operator for linear Rossby waves on a non-zonal flow, no theorems work [13]. Therefore, it is necessary to be extremely careful when generalizing the idea of a focal singularity to the case of Rossby waves. All this is due to the fact that the vertical mode of Rossby waves is extremely capricious and is no longer a blind follower of the ray in the horizontal plane.

Wave dynamics analysis Rossby in the presence of shear currents shows that regimes are possible when, due to the inhomogeneity of the background current, Rossby waves are captured by the current, while vertical focusing occurs i. e. compression of the mode at a certain vertical horizon. For the vertical mode, instead of the classical trigonometric cosine, strongly localized solutions appear in the form of exponentially modulated Hermite polynomials. Such solutions are well known for internal waves [14] and have also been constructed for Rossby waves [15].

The situation can be qualitatively described as follows. An inhomogeneous background current acts like a kind of parabolic antenna. The wave, falling into this parabolic trap, begins to reflect from the narrowing walls of the paraboloid, while the vertical transparency zone narrows and the movement of the wave towards the center of the paraboloid slows down more and more. The distance between adjacent points of reflection from the paraboloid mirror gradually decreases. This process in a linear setting lasts an infinitely long time. Since this result is obtained in the WKB approximation, a reference two-dimensional equation is additionally constructed. For internal waves, such an equation in the vicinity of the focus and its analysis were performed in [16]. Since there are no fundamental differences between Rossby waves and internal waves in the vicinity of the focus (the only difference is the estimation of dimensionless parameters), so it makes sense to present the mathematical part of the work for the already known reference equation for internal waves.

2. Statement of the problem. Two-dimensional reference equation. Baroclinic case

Consider a two-dimensional reference equation in the vicinity of the focus [16, equation 2.5]:

$$\Psi_{zz} + \left(\frac{y}{L_y} + \frac{z^2}{L_z^2}\right)\Psi_{yy} + \frac{2}{L_y}\Psi_y = 0, \tag{1}$$

where (x, y, z) is a rectangular coordinate system, Ψ — current function L_y and L_z are the lengths of inhomogeneities in y and z. Following the standard scheme, we will seek solutions localized in a small vicinity of some level along the vertical coordinate and exponentially decaying outside this level, where the following notation is introduced for the case of internal waves:

$$\frac{1}{L_y} = 2\nabla_y \ln\Omega, \quad \frac{1}{L_z^2} = \frac{\nabla_z^2\Omega}{\Omega} - \frac{\nabla_z^2 N}{N}, \tag{2}$$

where $\Omega = \omega - kU$, ω is a frequency, k is a zonal wavenumber, $U(z, y)$ is an inhomogeneous horizontal background shear flow, $N^2(z) = -g\frac{d}{dz}\ln\rho_0(z)$, $\rho_0(z)$ is a density. The value of all derivatives is taken at the focal point.

The capture of Rossby waves by a parabolic trap created by a flow in a stratified fluid can be qualitatively represented as follows (Fig. 1):

In the literature, it is often said that Fourier analysis does not work in inhomogeneous media, but this is not entirely true. Let us show that Fourier analysis can work, for example, on linear shear flows.

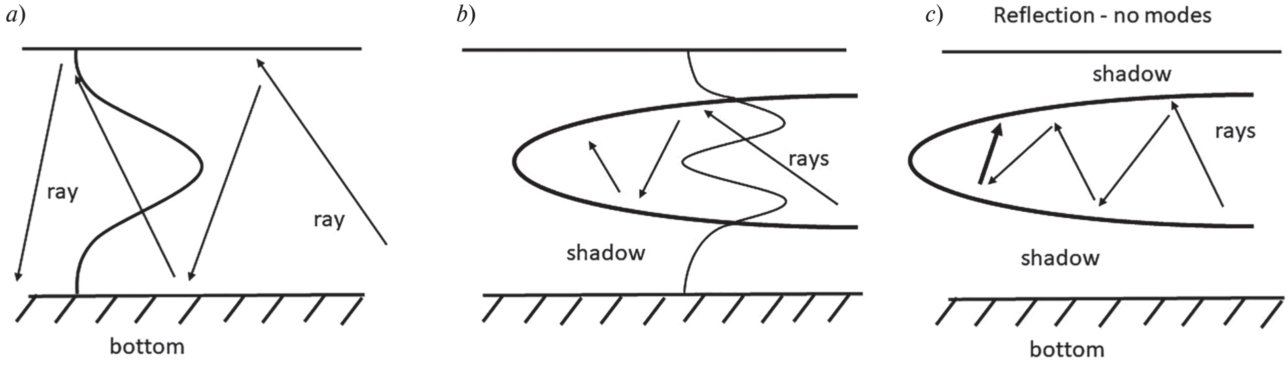

Fig. 1. Parabolic antenna: *a* — free wave mode in the absence of flow; *b* — mode at $L_z < 4L_y$ (interaction with the flow); *c* — reflection of waves from the flow at absence of modes (based on [16], as well as equation (13) of this work)

So, we seek the solution of Eq. (1) in the form of the Fourier integral, in this case we restrict ourselves to the beginning by the upper part of the integral

$$\Psi(k, y, z, \omega) = \int_0^\infty G(k, l, z, \omega) \exp(i\, l\, y)\, dl, \tag{3}$$

where l — meridional wave number, $G(k, l, z, \omega)$ is a Fourier transform. Taking the whole integral or only the upper (lower) part of the integral is actually a very important question, the discussion of which we will postpone until the final part of the work. We use the properties of the Fourier transform:

$$\Psi \to G, \quad \Psi_y \to i\,l\,G, \quad \Psi_{yy} \to -l^2 G, \quad y\Psi_{yy} \to -i\left(l^2 G\right)_l. \tag{4}$$

Substituting expansion (3) into equation (1) and, taking into account (4), for the Fourier transform G we obtain the following equation:

$$G_{zz} - \frac{l^2 z^2}{L_z^2} G - i\frac{l^2}{L_y} G_l = 0. \tag{5}$$

Equation (5) is not a separable equation. To make it so, we perform the following change of variables $(z, l) \to (\eta, \varphi)$, where

$$\eta = \frac{z\, l^{1/2}}{L_z^{1/2}}, \quad \varphi = l. \tag{6}$$

The Jacobian of the replacement is

$$\frac{\partial(\eta, \varphi)}{\partial(z, l)} = l^{1/2}. \tag{7}$$

Note that equations (6) and (7) contain $l^{1/2}$. Technically, it is this fact that requires considering only one of the parts of the Fourier integral. For simplicity, we first chose the upper, positive part of the integration to remove the square root question.

The question arises: why should one choose such a change of variables? The answer is contained in [14], where a solution was constructed in the WKB approximation. In fact, no reasoning about self-similarity is needed, since in a certain sense all "self-similarity" is reduced to a simple change of variables of the form (6).

In new variables (η, φ) equation (5) takes the form of an equation with separable variables:

$$G_{\eta\eta} - \eta^2 G - i\frac{\eta L_z}{2L_y} G_\eta - i\frac{\varphi L_z}{L_y} G_\varphi = 0. \tag{8}$$

Next, we look for a separable solution in the following form:

$$G(\eta, \varphi) = H(\eta) F(\varphi). \tag{9}$$

For $H(\eta)$ we obtain the following equation

$$H_{\eta\eta} - i\frac{\eta L_z}{2L_y} H_\eta - \left(\eta^2 + \mu_0\right) H = 0, \tag{10}$$

where μ_0 is a constant separation. Further, the term with the first derivative in equation (10) is removed by the following replacement

$$H(\eta) = P(\eta) \exp\left(i\frac{L_z}{8L_y}\eta^2\right). \tag{11}$$

For function $P(\eta)$ we obtain the following equation

$$P_{\eta\eta} + \left[-\eta^2\left(1 - \frac{L_z^2}{16L_y^2}\right) - \mu_0 + i\frac{L_z}{4L_y}\right] P = 0. \tag{12}$$

Recall that we are looking for solutions localized in the vicinity of the level ($z = 0$). Equation (12) shows that the coefficient at η^2 must be positive, therefore, we obtain the following condition for the existence of localized solutions

$$\left(1-\frac{L_z^2}{16L_y^2}\right)>0 \Leftrightarrow 0<\left|L_z\right|<4\left|L_y\right|. \tag{13}$$

Condition (13) says that the branches of the parabola, which limits the inner region of transparency from the outer region of the shadow, should be practically parallel to each other (Fig. 2). Otherwise, the vertical mode will not form, and the wave will not approach the critical point for an infinitely long time. Note that if condition (13) is not satisfied, then formally other modes of transformation of the solution are also possible.

Estimation of the parameters for internal waves shows that if we take the scales that are used in [16], then a very good separation of these values is obtained ($L_z < 4L_y$), and this indicates that the concept of a parabolic trap is justified from the point of view of physics.

Let us define the quantum values of the separation variable μ_0 (see [6, 17]):

$$-\left(2m+1\right)=\left(\mu_0-i\frac{L_z}{4L_y}\right)\Big/\left(1-\frac{L_z^2}{16L_y^2}\right)^{1/2}, \quad m=0,1,2,... \tag{14}$$

from where we find the eigenvalues:

$$\mu_0=\frac{L_z}{L_y}\left[\frac{1}{4}i-\frac{\delta}{2}\left(m+\frac{1}{2}\right)\right]; \quad \delta\equiv\left(\frac{16L_y^2}{L_z^2}-1\right)^{1/2}, \quad m=0,1,2,... \tag{15}$$

and own functions:

$$P\left(\eta\right)=\left[\sum_{m=0}^{\infty}H_m\left(\eta\left[1-\frac{L_z^2}{16L_y^2}\right]^{1/4}\right)\right]\exp\left[-\frac{\eta^2}{2}\left(1-\frac{L_z^2}{16L_y^2}\right)^{1/2}\right], \quad m=0,1,2,..., \tag{16}$$

where H_m — Hermite polynomials.

We now turn to the definition of the second factor $F(\varphi)$ in solution (9). From formula (8) we obtain the following equation:

$$-i\frac{\varphi L_z}{L_y}F_\varphi+\mu_0F=0. \tag{17}$$

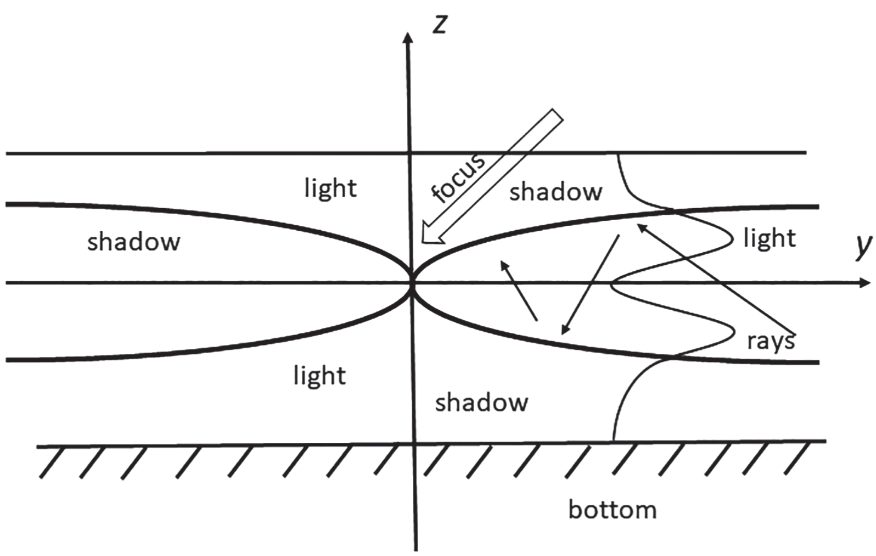

Fig. 2. Parabolic trap: antisymmetry of transparency and shadow areas at $L_z < 4L_y$.

The solution to equation (17) has the following form:

$$F(\varphi) = \varphi^{\mu}, \qquad \mu \equiv -i\mu_0 \frac{L_y}{L_z}. \tag{18}$$

Finally, we get the following eigenvalues:

$$\mu = \frac{1}{4} + i\frac{\delta}{2}\left(m + \frac{1}{2}\right). \tag{19}$$

Substituting all found composite solutions into the original integral (3), we find the eigenfunctions:

$$\Psi(k, y, z, \omega) = A(k, \omega) \sum_{m=0}^{\infty} \int_0^{\infty} l^{\mu} \left[H_m \left(\frac{z\, l^{1/2}}{L_z^{1/2}} - \left[1 - \frac{L_z^2}{16 L_y^2}\right]^{1/4} \right) \right] \times$$
$$\times \exp\left[-\frac{z^2 l}{2L_z} \left(1 - \frac{L_z^2}{16L_y^2}\right)^{1/2} \right] \cdot \exp\left[i\, l \left(y + \frac{z^2}{8L_y} \right) \right] dl, \tag{20}$$

where $A(k, \omega)$ — some constant that determines the spectral density of the initial state. Further, the obtained eigenfunctions (20) can be reduced using simple transformations to a degenerate hypergeometric function of some complex argument (Appendix). Note that it is the integral notation (20) that is preferable for finding the asymptotics of eigenfunctions. Although the constructed eigenfunctions (20) are functions of two physical variables (z and y), the integral for the eigenfunctions is one-dimensional, which makes it possible to use the stationary phase method [6].

We write out the imaginary part of the integral (20) in the following form:

$$\exp\left[i\, l \left(y + \frac{z^2}{8L_y} \right) + i\frac{\delta}{2}\left(m + \frac{1}{2}\right) \ln l \right]. \tag{21}$$

Differentiating (21) concerning the variable l and equating the expression in square brackets to zero, we obtain the equation for the point l_c.

$$y + \frac{z^2}{8L_y} = -\frac{\delta}{2l_c}\left(m + \frac{1}{2}\right). \tag{22}$$

Next, we rewrite the equation (22) in the following form:

$$l_c = -\frac{\delta\left(m + \frac{1}{2}\right)}{2\left(y + \frac{z^2}{8L_y}\right)}. \tag{23}$$

It is easy to see that the obtained equation (23) is a kind of generalization of the short-wavelength WKB asymptotics of the dispersion relation $l_c = y^{-1}$. Then the second derivative of the phase with respect to the wave number is proportional to l_c^{-2}, therefore, the root to the minus of the first power of the second derivative is proportional to l_c^{1}. Finally, the asymptotics of the eigenfunctions in the vicinity of the critical point will have the following form:

$$\Psi_1(k, y, z, \omega) = A(k, \omega) \sum_{m=0}^{\infty} l_c^{\mu+1} \left[H_m \left(\frac{z\, l_c^{1/2}}{L_z^{1/2}} \left[1 - \frac{L_z^2}{16L_y^2}\right]^{1/4} \right) \right] \cdot \exp\left[-\frac{z^2 l_c}{2L_z} \left(1 - \frac{L_z^2}{16L_y^2}\right)^{1/2} \right] \cdot \exp\left[i\frac{\delta}{2}\left(m + \frac{1}{2}\right) \right]. \tag{24}$$

Analyzing asymptotics (24), we can say the following: the asymptotics of the solution to the reference equation exactly coincides with the WKB solution with the vertical mode in the form of Hermite polynomials, majorized by the Gaussian function [14], and gives the classical power of 5/4 for the amplitude of the vertical velocity. If in [16] they talk about a certain mode, then we argue that this is not their vertical mode in the form of a WKB solution along the vertical coordinate, but a completely different mode, which was constructed in [14]. The constructed solutions are functions not of the variables (z, y), but of some curvilinear variables, which have the following form:

$$(y,z)\to\left(\left(y+\frac{z^2}{8L_y}\right),\ \frac{z}{\left(y+\frac{z^2}{8L_y}\right)^{1/2}}\right). \tag{25}$$

Thus, in a sense, there is a curvature of space in the vicinity of the focal point.

Note that this "curvilinearity" was also seen when solving the problem in the WKB approximation, where the following change of variables was formally performed: $(y,z)\to\left(y,\frac{z}{\sqrt{y}}\right)$. Therefore, by and large, the asymptotics of one-dimensional integrals do not give any qualitatively new results different from WKB-solutions, except for condition (13), which is satisfied with a large margin.

3. Discussion and conclusions

So, the first result in this work is the mathematically correct matching of the asymptotics of the reference solution in the far zone with the WKB solution in terms of the Hermite polynomials in the vertical coordinate. The solution of the reference equation in terms of a degenerate hypergeometric function of a certain complex variable [16] is matched not with the asymptotics in the far-field in terms of the WKB solution in the vertical coordinate, but with the asymptotics from another paper [14]. Although this result is academic, it can be useful in other areas of theoretical physics where such reference solutions are used.

It is shown that the question of the duality of the Fourier integrals cannot be a justification for the unique character of the strict absorption of the wave at the critical point. Another solution is not mathematically forbidden — the lower part of the integral. Consequently, the search for a solution in the form (3), as half of the Fourier integral, rather refers to a particular formulation of the problem. You can construct the solution as the lower half of the integral, or you can add both sides, which is also mathematically correct. In our opinion, this duality is reduced only to the question: what part of the Fourier integral should be taken. Everything is determined only by the specific formulation of the problem, namely by the boundary conditions, without which a specific solution cannot be obtained. Therefore, it is formulated as follows: can a wave fall on a critical point for an infinitely long time without reflection? The answer is "Yes". It is precisely such a solution that was constructed in [16], which we obtained in this work in an alternative way. But it would be more customary to write the sign Re in front of the integral (3), that is, to single out the real part of the solution. That is, it all comes down to the addition of the solution found in [16] in a complex variable with a solution with a complex conjugate argument. However, such an action will lead to a solution in the form of a sum of two solutions, which will no longer be an infinitely long incident on the focus wave. Consequently, other formulations of the problem are possible, which will lead to a different answer, since there is no mathematical prohibition on other formulations of the problem. That is, it all comes down to the addition of the solution found in [16] in a complex variable with a solution with a complex conjugate argument. However, such an action will lead to a solution in the form of a sum of two solutions, which will no longer be an infinitely long incident on the focus wave. Consequently, other formulations of the problem are possible, which will lead to a different answer, since there is no mathematical prohibition on other formulations of the problem. That is, it all comes down to the addition of the solution found in [16] in a complex variable with a solution with a complex conjugate argument. However, such an action will lead to a solution in the form of a sum of two solutions, which will no longer be an infinitely long incident on the focus wave. Consequently, other formulations of the problem are possible, which will lead to a different answer, since there is no mathematical prohibition on other formulations of the problem.

We have shown that the primary quantization of the reference function is the classical quantization of the vertical Sturm-Liouville problem. Secondary quantization is the classical point of the stationary phase. Note that the course of solutions shows how the 5/4 degree is added. Secondary quantization gives a degree of 1, this is the degree of a barotropic problem. Taking into account the baroclinicity leads to a new parameter determined by the formula (19), from which it can be seen that the baroclinicity gives an additional contribution of 1/4. This is the contribution to the amplification of the amplitude factor due to the vertical compression of the wave. Thus, the overall total amplification of amplitude fluctuations is made up of two factors: the horizontal splitting of the wave, which gives a degree of 1, and additional vertical compression of the wave, which gives a degree of 1/4, and the total is 5/4.

Note that in this work, with the second quantization, one can restrict oneself only to the stationary phase method, and this is due to the fact that the wave is quasi-monochromatic. However, for a more complete analysis, the obtained solution still needs to be convoluted in two components — the frequency and the longitudinal component of the wavenumber, and in this case, the saddle-point method is already needed. This analysis will be presented in a separate work.

Another result is the question of transmission coefficient. For a one-dimensional function, the reference solution has a classical logarithmic branch point, which gives an exponential transmission coefficient, and here we can agree with [16]. However, in reality, for a full-fledged two-dimensional function, the question of passing the pole is much more complicated. If you look at Figure 2, it becomes clear that in the two-dimensional case, in contrast to the one-dimensional case, there is a mirror symmetry of the transparency and shadow regions before and behind the focus. Therefore, a qualitatively new analysis is required to study the problem of transmission, which is fundamentally different from the one-dimensional approach. If in this work we were able to stitch two-dimensional functions in the far zone from the side of the wave incidence on the focus, then stitching two-dimensional functions behind the shadow region is a question, which, as far as we know, has not yet been considered by anyone in a two-dimensional setting, and it requires the development of completely new approaches. Figure 2 demonstrates that if the solution is stitched together as a one-dimensional function, then there will be an exponentially decaying tail behind the focus, and the transmission coefficient will be exponentially small. However, it can be noted that for a two-dimensional solution, the mode can go not only to the shadow region but also to the transparency regions behind the focus. Therefore, on a qualitative level, it is clear that the one-dimensional result of an exponentially small transmission is greatly exaggerated. The wave can overcome the focus by rearranging its mode along the vertical, and thus the question of the transmission coefficient is rather open.

Due to the large horizontal scales, the analysis of Rossby waves in the vicinity of the focal point requires consideration of the question: how stable is the result of infinite incidence on the focus when additional parameters such as topography and stratification are taken into account. Analysis of the problem in a linear formulation about the possible joint influence of topography and barotropic variable shear flow suggests that, rather, these factors are widely separated in scale, and their combined effect is negligible [17–19]. However, in our opinion, taking into account stratification in works such as [7, 8] is greatly underestimated.

The main result of this work is that, in the linear setting, the paraboloid mirror should change rather smoothly so that the wave does not leave the process of infinitely long incidence onto the focus. However, in the regions of strong jet flows, the WKB approximation and the linear formulation certainly do not work, and then the consideration of the nonlinear formulation is required. The nonlinear critical layer of Rossby waves is more a reflector than an absorber, and the nonlinearity can lead to the effect of a waveguide [19, 20] All these facts indicate the need for an extremely careful approach in the analysis of Rossby waves.

4. Appendix. Reduction of the Fourier Integral to a Hypergeometric Function of a Complex Variable

For comparison with the solution [16], we rewrite the eigenfunctions (20) in the following form:

$$\Psi_m(k,y,z,\omega)=\int_0^\infty l^\mu H_m\left(\frac{z\,l^{1/2}}{2L_y^{1/2}}\delta^{1/2}\right)\exp\left[-\frac{z^2 l}{8L_y}\delta\right]\cdot\exp\left[i\,l\left(y+\frac{z^2}{8L_y}\right)\right]dl. \tag{26}$$

Let's make the change of variables ($l\to x$) Argument $\left(\frac{z\,l^{1/2}}{2L_y^{1/2}}\delta^{1/2}\right)$ the Hermite polynomial is taken as a new variable

$$x=\frac{zl^{1/2}\delta^{1/2}}{2L_y^{1/2}}. \tag{27}$$

Hence,

$$\Psi_m\sim\int_0^\infty \exp\left[-2ax^2\right]\frac{x^{2\mu+1}}{z^{2\mu+2}}H_m(x)\,dx. \tag{28}$$

Note: in formula (28) a complex variable (2a) appeared, depending on two spatial physical variables (z,).

$$2a = \frac{1}{2} - i\frac{1}{2\delta z^2}\left(z^2 + 8\,y\,L_y\right). \tag{29}$$

In fact, the two-dimensional problem was solved through a one-dimensional integral, but only from a complex argument. It is not hard to see that

$$\frac{1}{2a} = \frac{2\delta z^2}{\delta z^2 - i\left(z^2 + 8\,y\,L_y\right)} \equiv \tau^*, \tag{30}$$

where τ — complex variable [16], * — complex conjugation. The integral representation of the hypergeometric function in terms of the Hermite polynomials has the form (see [21], sections 7.37–7.38),

which is the formula 7.376.2, p. 852:

$$F\left(-n; \frac{\nu+1}{2}; \frac{1}{2}; \frac{1}{2a}\right) \sim \int_0^\infty \exp\left[-2ax^2\right] x^\nu H_{2n}(x)\,d\,x, \tag{31}$$

where Re$a > 0$, Re$v > -1$;

which is the formula 7.376.3, p. 852:

$$F\left(-n; \frac{\nu}{2}+1; \frac{3}{2}; \frac{1}{2a}\right) \sim \int_0^\infty \exp\left[-2ax^2\right] x^\nu H_{2n+1}(x)\,d\,x, \tag{32}$$

where Re$a > 0$, Re$v > -2$.

Taking into account (15), we find:

$$\nu = 2\mu + 1 = \frac{3}{2} + i\delta\left(m + \frac{1}{2}\right). \tag{33}$$

Consequently, the constructed solutions are regular, the integrals converge. The similarity with the solution [16] was achieved in three out of four parameters. Determine the last parameter of the hypergeometric function:

$$\frac{\nu}{2} + 1 = \frac{7}{4} + i\frac{\delta}{2}\left(m + \frac{1}{2}\right) \equiv \gamma^*, \tag{34}$$

where γ Is the quantum parameter from [16] (formula (2.7)). Find the second parameter in a similar way:

$$\frac{\nu+1}{2} = \frac{5}{4} + i\frac{\delta}{2}\left(m + \frac{1}{2}\right). \tag{35}$$

Thus, we got full compliance with the work [16]. If we take into account the second part of the Fourier integral for negative wavenumbers, then by changing the variable it can be reduced to an integral over positive wavenumbers, while in the integral under study, the imaginary unit will be replaced ($i \to -i1$), and this will lead to the fact that the second part of the solution appears in which instead of τ and γ will τ^* and γ^*. Thus, the general solution to the problem is the sum of solutions from τ and τ^*, which is physically equivalent to the sum of the incident and reflected waves. Therefore, mathematically, there is no prohibition on reflection, and the version of infinite focusing is greatly exaggerated. When the wave slows down more and more as it moves towards the focus, it is physically difficult to understand whether it is an incident wave or a standing mode, therefore, in the applied sense, there is not a big difference between half of the Fourier integral solution (3) or the whole Fourier integral, when the integration limits are from $-\infty$ to $+\infty$, which is equivalent to writing the sign Re in front of the integral (3).

5. Funding

The study was carried out with the financial support of the Russian Foundation for Basic Research within the framework of scientific project No. 20–05–0006. The work of V.G. Gnevyshev was also carried out within the framework of the state assignment of the Institute of Oceanology of the Russian Academy of Sciences No. 0128–2021–0003.

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