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МОДЕЛИ НЕЛИНЕЙНЫХ ДЛИННЫХ ВНУТРЕННИХ ВОЛН ВО ВРАЩАЮЩЕМСЯ ОКЕАНЕ

Известное уравнения Кортевега-де Вриза, применяемое для описания длинных нелинейных внутренних волн в присутствии вращения Земли, заменяется уравнением Островского. Здесь мы даем асимптотический вывод этого уравнения, учитывая фоновое сдвиговое течение и плотностную стратификацию. Затем обобщаем эту модель, чтобы учесть горизонтальную неоднородность параметров среды, и описываем, как начальный солитон уравнения Кортевега-де Вриза деформируется и излучает инерционно-гравитационные волны.

Ключевые слова: нелинейные внутренние волны, вращающийся океан, Кортевег-де Вриз подобные уравнения.

Introduction. It is well known that the internal solitary waves commonly observed in the coastal ocean can be modelled by the Korteweg–de Vries (KdV) equation, or a related equation, see the reviews by [1] and [2] for instance. When expressed in a reference frame moving with the linear long wave speed c, the KdV equation is

$$A_t + \mu A A_x + \lambda A_{xxx} = 0. \tag{1}$$

Here $A(x, t)$ is the amplitude of the linear long wave mode $\phi(z)$ corresponding to the linear long wave phase speed c, which is determined from the modal, see Eqs.(14), (15). The coefficients μ and λ are given by certain integrals involving the modal function, see (17), (18).

However, oceanic internal waves are often observed to propagate for long distances over several inertial periods, and hence the effect of the Earth's background rotation needs to be taken into account. Although the effect of this background rotation is small for an individual wave, it is potentially significant for the wave evolution. The simplest model equation which takes account of the background rotation is the Ostrovsky equation, which is an adaptation of the KdV Eq.(1), derived for water waves by [3] and for internal waves by [4], and given by,

$$\{A_t + \nu A A_x + \lambda A_{xxx}\}_x = \gamma A, \tag{2}$$

The background rotation is represented by the coefficient γ, which in the absence of a background shear flow, is given by

$$\gamma = \frac{f^2}{2c}, \tag{3}$$

where f is the Coriolis parameter. The explanation for the origin of this extra term lies in the linear dispersion relation for long waves, which is $\omega^2 \approx c^2 k^2 + f^2$ for the frequency ω

and wavenumber k. The dominant balance is $\omega \approx kc$, and moving to the reference frame moving with speed c and adding the next order cubic linear dispersive correction, leads to the KdV equation. When weak rotation is added, this becomes $\omega \sim kc + f^2/2kc$, and so leads to the Ostrovsky equation.

For oceanic internal waves $\lambda\gamma > 0$, see (3) and (18), and then it is known that Eq.(16) does not support steady solitary wave solutions, see [5–8] and the references therein. The simplest explanation is that the additional term on the right-hand side of (2) removes the spectral gap on which solitary waves exist for the KdV equation. Thus, the linear dispersion relation of the Ostrovsky Eq.(2) for the phase velocity c_p as a function of wavenumber k is given by

$$c_p = \frac{\gamma}{k^2} - \lambda k^2 .$$

For the KdV Eq.(1) ($\gamma = 0$) there is a gap in the spectrum for all $c_p > 0$ where solitary waves can exist. But there is no such gap for the Ostrovsky equation, and hence no solitary waves are expected to occur are expected to occur since $\lambda\gamma > 0$ for internal waves. Further [8–10] have shown that the long-time effect of rotation is the destruction of the initial internal solitary wave by the radiation of small-amplitude inertia-gravity waves, and the eventual emergence of a coherent steadily propagating nonlinear wave packet.

In this review paper we re-examine the derivation of the Ostrovsky equation, incorporating a background shear flow as well as the background stratification. We describe how the model can be extended to a variable-coefficient Ostrovsky equation to take account of horizontal variability in the background fields and we describe the asymptotic slowly varying solitary wave solution, and demonstrate that the solitary wave is again eventually destroyed in a variable medium as in a medium with a uniform background state.

Formulation and derivation. We assume that the fluid is in inviscid and incompressible, and in the basic state has depth h, a density stratification $\rho_0(z)$ and a horizontal shear flow $u_0(z)$ in the x-direction. Further we assume that the flow is two-dimensional, and so all variables depend only on x, z where z is the vertical co-ordinate, and the time t. Using the long-wave variables

$$X = \varepsilon x,\ T = \varepsilon t, \tag{4}$$

the dependent perturbation variables are then the velocity field $(u, \varepsilon v, \varepsilon\omega)$ the pressure p and density ρ, and the auxiliary variable ζ which is the vertical particle displacement. The Coriolis parameter is $\varepsilon^2 f$, scaled to reflect that rotation is a weak effect. Then the full equations for these perturbation variables are, expressed to the required order,

$$\begin{aligned}&\rho_0(u_T + u_0 u_X + \omega u_{0z}) + p_X = F_1\\ &= -\rho_0(uu_X + \omega u_z) - \rho(u_T + u_0 u_X + \omega u_{0z}) + \varepsilon^2(\rho_0 + \rho) fv,\end{aligned} \tag{5}$$

$$\rho_0(v_T + u_0 u_X + fu) + \rho f u_0 = \dots . \tag{6}$$

$$p_z + g\rho = -\varepsilon^2 \rho_0(\omega_T + u_0\omega_X) + \dots , \tag{7}$$

$$u_X + \omega_z = 0, \tag{8}$$

$$\zeta_T + u_0\zeta_X - \omega = J_1 = -u\zeta_X - \omega\zeta_z, \tag{9}$$

$$\rho_t + u_0\rho_x + \omega\rho_{0z} = -u\rho_X - \omega\rho_z, \tag{10}$$

and the boundary conditions are

$$\omega = 0, \quad \text{at} \quad z = -h, \tag{11}$$

$$p + p_0 = 0, \quad \text{at} \quad z = \eta,$$

$$\zeta = \eta \quad \text{at} \quad z = \eta.$$

The density Eq.(10) can be solved by $\rho_0(z) + \rho = \rho_0(z - \zeta)$, so that

$$\rho = -\rho_{0z}\zeta + \frac{\rho_{0zz}\zeta^2}{2} + \dots .$$

Then the vertical momentum Eq.(7) becomes

$$\rho_z + \rho_0 N^2 \zeta = G_1 = -\frac{g\rho_{0zz}\zeta^2}{2} - \varepsilon^2 \rho_0 (\omega_T + u_0 \omega_X) + \dots ,$$

where $\rho_0 N^2 = -g\rho_{0z}$.

The free surface boundary conditions (13), (14) are expanded so that

$$p - g\rho_0\eta = g\rho\eta + \frac{g\rho_{0z}\eta^2}{2} + \dots , \quad \zeta + \eta\zeta_z + \dots = \eta, \quad \text{at} \quad z = 0,$$

and can then be combined to give

$$p - g\rho_0\zeta = H_1 = \frac{\rho_0 N^2 \zeta^2}{2} + g\rho_0\zeta\zeta_z + \dots , \quad \text{at} \quad z = 0. \tag{12}$$

Thus the variables ρ, η are formally eliminated.
Next we change variables

$$s = X - cT, \qquad \tau = \varepsilon^2 T.$$

Then the system (5)–(10) becomes

$$\rho_0(-Wu_s + \omega u_{0z}) + p_s = \tilde{F}_1 = F_1 - \varepsilon^2 \rho_0 u_\tau, \;\; \rho_0(Wv_s + fu) + \rho f u_0 = \dots .$$

$$p_z + \rho_0 N^2 \zeta = \tilde{G}_1 = -\frac{g\rho_{0zz}\zeta^2}{2} + \varepsilon^2 \rho_0 W \omega_s \;, \;\; u_s + \omega_z = 0, \;\; -W\zeta_s - \omega = \tilde{J}_1 = J_1 - \zeta_\tau.$$

Here $W = c - u_0$. The boundary conditions (11), (12) are unchanged. We then seek a solution in the form

$$[u, v, \omega, p, \zeta] = \varepsilon^2 [u_1, v_1, \omega_1, p_1, \zeta_1] + \varepsilon^4 [u_2, v_2, \omega_2, p_2, \zeta_2] + \dots .$$

At the leading order the solution is

$$u_1 = A(W\phi)_z, \; \omega_1 = -A_s W\phi, \; p_1 = A\rho_0 W^2 \phi_z, \; \zeta_1 = A\phi,$$

$$v_1 = fB\Phi, \;\; \rho_0 W\Phi = \rho_0 W\phi_z - (\rho_0 u_0)_z \phi, \quad B_s = A. \tag{13}$$

Here the modal function $\phi(z)$ satisfies the modal system,

$$(\rho_0 W^2 \phi_z)_z + \rho_0 N^2 \phi = 0, \tag{14}$$

$$\phi = 0 \quad \text{at} \quad z = 0, \qquad \text{and} \qquad W^2 \phi_z = g\phi \quad \text{at} \quad z = 0. \tag{15}$$

At the next order, we get the system

$$\rho_0(-Wu_{2s}+\omega_2 u_{0z})+p_{2s}=\tilde{F}_1,$$

$$p_{2z}+\rho_0 N^2\zeta_2=\tilde{G}_1,$$

$$u_{2s}+\omega_{2z}=0,$$

$$-W\zeta_{2s}-\omega_2=\tilde{J}_1.$$

while the boundary conditions are $\omega_2=0$, at $z=-h$, $p_2-g\rho_0\zeta_2=H_1$, at $z=0$, where $F_1,\ldots$ etc. are evaluated at the leading order solution, $u_1,\ldots$. Note that eliminating u_2 yields $\rho_0(\omega_2 u_{0z}+W\omega_{2z})+p_{2s}=\tilde{F}_1$, and then eliminating ω_2 gives

$$-\rho_0 W^2\zeta_{2sz}+p_{2s}=F_2=\tilde{F}_1+\rho_0(u_{0z}\tilde{J}_1+W\tilde{J}_{1z}).$$

Finally eliminating p_2 yields $(\rho_0 W^2\zeta_{2sz})_z+\rho_0 N^2\zeta_{2s}=I_2=\tilde{G}_{1s}-F_{2z,}$ with the boundary conditions $W\zeta_{2s}=0$ at $z=-h$, $\rho_0 W^2\zeta_{2sz}-g\rho_0\zeta_{2s}=H_2=H_{1s}-F_2$ at $z=\eta_0$.

The compatibility condition is

$$\int_{-h}^{0} I_2\phi dz-[\rho_0\phi H_2](z=0)=0.$$

This can be written in the form

$$\int_{-h}^{0}\tilde{G}_{1s}\phi dz+\int_{-h}^{0}F_2\phi_z dz-[\rho_0 H_{1s}\phi](z=0)=0.$$

Here

$$\tilde{G}_{1s}=(\rho_0 N^2)_z\phi^2 AA_s-\rho_0 W^2\phi A_{sss,}$$

$$F_2=-2\rho_0 W\phi_z A_\tau+(-\rho_0[2WW_z\phi\phi_z+W^2\phi_z^2-W^2\phi\phi_{zz}]-\rho_{0z}W^2\phi\phi_z)AA_s+f^2 B\rho_0\Phi,$$

$$H_{1s}=(\rho_0 N^2\phi^2+2\rho_0 W^2\phi_z^2)AA_s.$$

Making the substitutions yields the Ostrovsky equation

$$A_t+\mu AA_s+\lambda A_{sss}=\gamma B,\ B_s=A, \text{ that is } \{A_\tau+\mu AA_s+\lambda A_{sss}\}_s=\gamma A. \tag{16}$$

where the coefficients are given by

$$I\mu=3\int_{-h}^{0}\rho_0 W^2\phi_z^2 dz, \tag{17}$$

$$I\lambda=\int_{-h}^{0}W^2\phi^2 dz, \tag{18}$$

$$I\gamma=f^2\int_{-h}^{0}\rho_0\Phi\phi_z dz. \tag{19}$$

$$I = 2\int_{-h}^{0} \rho_0 W \phi_z^2 dz.$$

When $f = 0$ this reduces to the usual KdV equation. Note that the expressions for the coefficients μ, λ are the same as those when the rotation is zero. The expression (19) for γ extends the expression (3) to the case when there is a background shear flow $u_0(z)$. Note that if $u_0 = 0$, then $\Phi = \phi_z$, and so $\gamma = f^2/2c$ as in (3). An alternative derivation of (19) from the linear modal equation extended to take account of rotation is presented below. Whereas $\gamma\lambda > 0$ when there is no shear flow, this may not be always be the case when the shear flow is present, and this issue is examined below.

Variable depth and hydrography. When the bottom topography and hydrography vary slowly in the *x*-direction the Ostrovsky Eq.(16) is replaced by the variable-coefficient Ostrovsky equation

$$\{A_T + cA_X + \frac{cQ_X}{2Q}A + \mu AA_X + \lambda A_{XXX}\}_X = \gamma A. \tag{20}$$

Here as above $A(X, T)$ is the amplitude of the wave, and X, T are space and time variables again defined by (4). The coefficients, c, μ, λ, γ are defined as above, while Q is the linear magnification factor, given by

$$Q = Ic^2.$$

It is defined so that $Q\eta^2$ is the wave action flux. Each of these are slowly varying functions of X. In the absence of rotation, Eq.(20) was derived by [11] for water waves and by [12] for internal waves (for a recent review, see [13]). The derivation assumes the usual KdV and Ostrovsky equation balance as described above, and in addition assumes that the waveguide properties (i.e. the coefficients c, Q, μ, λ) vary slowly so that Q_x/Q for instance is of the same order as the dispersive and nonlinear terms.

The first two terms in (20) are the dominant terms, and it is then useful to make the transformation

$$A = \sqrt{Q}\eta, \qquad t = \int^{\chi} \frac{d\chi}{c}, \qquad x = t - \tau. \tag{21}$$

Substitution into (20) yields, to the same order of approximation as in the derivation of (20),

$$\{A_t + \alpha AA_x + \delta A_{xxx}\}_x = \beta A, \qquad \alpha = \frac{\mu}{c\sqrt{Q}}, \qquad \delta = \frac{\lambda}{c^3}, \qquad \beta = \gamma c = \frac{f^2}{2}.$$

Here the coefficients α, δ, β are functions of t alone. Note that although t is a variable along the spatial path of the wave, we shall subsequently refer to it as the “time”. Similarly, although x is a temporal variable, in a reference frame moving with speed c, we shall subsequently refer to it as a «space» variable.

Eq.(20) can be written as

$$A_t + \alpha AA_x + \delta A_{xxx} = \beta B, \;\; B_x = A. \tag{22}$$

Eq.(22) has two conservation laws for localised solutions

$$\int_{-\infty}^{\infty} A dx = 0\,, \tag{23}$$

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} A^2 dx = 0. \tag{24}$$

The first (23) is a zero-mass condition, and note that when A is localised, then so is B; indeed B, like A, also has zero mean. The second (24) expresses momentum conservation.

Slowly-varying solitary waves. We now suppose that the coefficients α, δ are slowly varying, and that β is small, and write

$$\alpha = \alpha(T), \qquad \delta = \delta(T), \qquad \beta = \varepsilon\tilde{\beta}, \qquad T = \varepsilon t, \qquad \varepsilon << 1.$$

We seek a standard multi-scale expansion for a modulated wave, namely

$$A = A^{(0)}(\theta, T) + \varepsilon A^{(1)}(\theta, T) + \ldots\,, \quad \theta = x - \frac{1}{\varepsilon}\int^{T} V(T)dT.$$

Substitution into (22) yields at the leading orders

$$-VA_\theta^{(0)} + \alpha A^{(0)} A_\theta^{(0)} + \delta A_{\theta\theta\theta}^{(0)} = 0,$$

$$-VA_\theta^{(1)} + \alpha(A^{(0)}A^{(1)})_\theta + \delta A_{\theta\theta\theta}^{(0)} = -A_T^{(0)} + \tilde{\beta}B^{(0)}, \quad B_\theta^{(0)} = A^{(0)}. \tag{25}$$

Each of these is essentially an ordinary differential equation with θ as the independent variable, and T as a parameter.

The solution for $A^{(0)}$ is taken to be the solitary wave

$$A^{(0)} = a\sec h^2(K\theta), \tag{26}$$

where $V = \dfrac{\alpha a}{3} = 4\delta K^2.$ (27)

At the next order, we seek a solution of (25) for $A^{(1)}$ which is bounded as $\theta \to \pm\infty$, and in fact $A^{(1)} \to 0$ as $\theta \to \infty$. The adjoint equation to the homogeneous part of (25) is

$$-VA_\theta^{(1)} + \alpha A^{(0)} A_\theta^{(1)} + \delta A_{\theta\theta\theta}^{(1)} = 0.$$

Two solutions are 1, $A^{(1)}$; while both are bounded, only the second solution satisfies the condtion that $A^{(1)} \to 0$ as $\theta \to \infty$. A third solution can be constructed using the variation-of-parameters method, but it is unbounded as $\theta \to \pm\infty$. Hence only one orthogonally condition can be imposed, namely that the right-hand side of (25) is orthogonal to $A^{(0)}$, which leads to

$$\frac{\partial}{\partial T}\int_{-\infty}^{\infty}[A^{(0)}]^2 d\theta = \tilde{\beta}[B^{(0)}]^2(\theta \to -\infty). \tag{28}$$

Note that $B^{(0)}$ $(\theta \to \infty) = 0$, and so

$$B^{(0)} = \int_{\infty}^{\theta} A^{(0)} d\theta = -\frac{a}{K}(1 - \tanh(K\theta)).$$

As the solitary wave (26) has just one free parameter (e.g. the amplitude a), this equation suffices to determine its variation. Substituting (26), (27) into (28) leads to the law

$$\hat{A}^{1/2}\hat{A}_T = -3\tilde{\beta}\{\frac{12\delta}{\alpha}\}^{2/3}\hat{A}, \qquad \hat{A} = \{\frac{12\delta}{\alpha}\}^{1/3} a.$$

This has the solution

$$\hat{A}^{1/2} = \hat{A}_0^{1/2} - \tilde{\beta} s, \qquad s = \int_0^T \{\frac{12\delta}{\alpha}\}^{2/3} dT.$$

Thus, as for constant depth case, the solitary wave is extinguished in finite time. For constant depth, the extinction time is, in dimensional coordinates

$$t_e = \frac{1}{\beta}\{\frac{\alpha a_0}{12\delta}\}^{1/2} = \frac{1}{\gamma}\{\frac{\mu\eta_0}{12\lambda}\}^{1/2}, \quad \chi_e = \frac{t_e}{c},$$

where here we recall that "time" is really "distance" along the path, that is, $t = \chi/c$ from (21). Note that we have assumed that $\delta > 0$, $\beta > 0$ which is the case for waves propagating in the positive ξ-direction, and have also assume for simplicity that $\alpha > 0$ and so $a > 0$. But if $\alpha < 0$, then $a < 0$ and we can simply replace δ, a with $|\delta|$, $|a|$ in these expressions.

We now recall that Eq.(22) has two conservation laws (23), (24). The condition (28) is easily recognized as the leading order expression for conservation of *local* momentum. But since this completely defines the slowly-varying solitary wave, we now see that this cannot simultaneously conserve total mass. This is apparent when one examines the solution of (25) for $A^{(1)}$, from which it is readily shown that although $A^{(1)} \to 0$ as $\theta \to \infty$, $A^{(1)} \to \beta\, B^{(0)}\, (\theta \to -\infty)\theta/V$ as $\theta \to -\infty$. This non-uniformity in the slowly-varying solitary wave has been recognized for some time, see, for instance, [14] and the references therein. The remedy is the construction of a trailing shelf A_s of small amplitude $O(\varepsilon)$ but long length-scale $O(1/\varepsilon)$, which thus has $O(1)$ mass, but $O(\varepsilon)$ momentum. It resides behind the solitary wave, and to leading order becomes trailing inertia-gravity waves. In the case of constant background environment, this trailing radiation develops into a nonlinear wave packet propagating with the maximum group velocity, see [8–10]. We would expect that a similar outcome may be also the case when the background varies, but that is a topic for future study:

$$\varepsilon A^{(1)} \sim -\frac{2\beta a\theta}{KV}, \text{ as } \theta \to -\infty. \tag{29}$$

To leading order, the trailing wave is a linear long wave. At the location of the solitary wave, it is given by

$$A \sim b\sin(k\theta), \text{ where } Vk^2 = \beta. \tag{30}$$

Here we assume that this trailing wave and the solitary wave must have the same phase θ in order to match. Hence k is found from the linear long wave dispersion relation, and is determined from V. Letting $k \to 0$ in (30) and matching with (29) yields

$$bk = -\frac{2\beta a}{KV}, \text{ so that } b = -\frac{6\beta}{Kk\alpha} = -\frac{12(\beta\delta)^{1/2}}{\alpha}, \;\; \eta_s = -\frac{12(\gamma\lambda)^{1/2}}{\mu}. \tag{31}$$

This determines the trailing wave amplitude b, or η_b in terms of the original variables. Note that it is proportional to $\beta^{1/2}$, and interestingly is independent of the solitary wave amplitude a, and has the opposite polarity. However, we see from (31) that since V is decreasing, the wavenumber k of the trailing radiation increases. Also, although the amplitude b is independent of a, it does vary with α, δ as the wave shoals.

In contrast, when $\beta = 0$, the expression for the solitary wave amplitude continues to hold, and yields the well-known expression

$$\hat{A} = \hat{A}_0, \qquad \frac{a}{a_0} = \left(\frac{\alpha\delta_0}{\alpha_0\delta}\right)^{1/3}, \qquad \frac{\eta}{\eta_0} = \left(\frac{\mu c^2 Q_0^2 \lambda_0}{\mu_0 c_0^2 Q^2 \lambda}\right)^{1/3}.$$

Also, the trailing shelf is determined in a different way, as now

$$A^{(1)} \to -\frac{M_T^{(0)}}{V}, \text{ as } \theta \to -\infty, \text{ where } M^{(0)} = \int_{-\infty}^{\infty} A^{(0)} d\theta.$$

Since $M^{(0)} = 2a/K$ is the mass of the solitary wave, this is just the expression of conservation or mass. Hence the amplitude of the trailing shelf at the solitary wave location is $A_s = A^{(1)}$ $(\theta \to -\infty)$, we find that

$$A_s = \frac{2a_T}{VK} = \frac{a_T}{4\delta K^3} = \frac{12^{1/2}\bar{\alpha}_T}{C_0^{1/2}\delta\bar{\alpha}^{5/3}}, \qquad \bar{\alpha} = \frac{\alpha}{\delta}, \qquad C_0 = \frac{a_0}{\bar{\alpha}_0^{1/3}}.$$

When $\bar{\alpha}$ is decreasing, A_s has the opposite polarity to the solitary wave. This can now be compared with the corresponding expression (31) when rotation is present.

Conclusion. Since oceanic internal solitary waves often survive for several inertial periods, it is necessary to replace the usual Korteweg–de Vries model (1) with the Ostrovsky Eq.(2). In this paper, we have reviewed the derivation of the Ostrovsky equation, and also extended it by allowing for a background shear flow as well as the background density stratification. Then we have presented a variable-coefficient Ostrovsky equation to take account of variable topography and hydrography, and suggest that this should be the basic model to describe the evolution of large amplitude oceanic internal waves. Its properties are yet to be determined in full, but here we show that, under the combined effect of rotation and variable background, a slowly varying solitary wave deforms adiabatically while being extinguished in finite time by the radiation of inertia-gravity waves. In a uniform medium [8–10] have shown that the initial solitary wave is replaced by another coherent structure. It has yet to be seen if the same, or similar, outcome will occur in a variable medium.

Appendix

Modal equation with rotation. An explanation of the form that γ (19) takes, can be found by considering the form of the modal Eq.(14) when rotation is included *a priori*,

$$\left(\rho_0 (W^2\phi_z - \frac{\varepsilon^2 f^2}{k^2}\Phi\right)_z + \rho_0 N^2 \phi = 0,$$

$$\phi = 0 \quad \text{at} \quad z = 0, \qquad \text{and} \qquad W^2\phi_z - \frac{\varepsilon^2 f^2}{k^2}\Phi = g\phi \quad \text{at} \quad z = 0.$$

Here k is the wavenumber, and we recall that $\varepsilon^2 f$ is the Coriolis parameter, while Φ is defined in (13). When $u_0(z) = 0$, $\Phi = \phi_z$ and this just replaces c^2 with $c^2 + \varepsilon^2 f^2/k^2$, or c with $c^2 + \varepsilon^2 f^2/2ck$, and then the usual explanation of the extra term added to the KdV equation t_0 produce the Ostrovsky equation follows. But in the presence of a shear flow, this argument cannot be used. Instead treat $\varepsilon^2 f^2/k^2$ as a small perturbation, and expand c as $c + \delta c$. To find δc, note that

$$\int_{-h}^{0}\left(\rho_0 (W^2\phi_z^2 - \frac{\varepsilon^2 f^2}{k^2}\Phi\phi_z\right)dz = \int_{-h}^{0}\rho_0 N^2\phi^2 dz + [\rho_0 g\phi]_{z=0}.$$

This yields

$$2\delta c\int_{-h}^{0}\rho_0 W\phi_z^2 dz = I\delta c = \frac{\varepsilon^2 f^2}{k^2}\int_{-h}^{0}\rho_0\Phi\phi_z dz,$$

which agrees with (19).

Examples. It is now of interest to find the sign of γ (19), assuming that $c > \max[u_0(z)]$, $W > 0$, and so $I > 0$. If $u_0 = 0$, then $\gamma = f^2/2c > 0$. The question is then, can shear cause $\gamma < 0$? In general

$$\begin{aligned} I\gamma &= f^2\int_{-h}^{0}\left\{\rho_0\phi_z^2 - \frac{(\rho_0 u_0)_z}{W}\phi\phi_z\right\}dz,\\ &= f^2\left\{\int_{-h}^{0}\rho_0\phi_z^2 dz - [\phi^2(\frac{\rho_0 u_{0z}}{2W})]_{z=0} + \int_{-h}^{0}\phi^2\left[\frac{(\rho_0 u_0)_z}{2W}\right]_z dz\right\}, \qquad (32)\\ &= f^2\left\{\int_{-h}^{0}\phi^2\left[\frac{\rho_0 N^2}{W^2} + \frac{3(\rho_0 u_{0z})_z}{2W}\right]dz + \left[\phi^2\left(\frac{\rho_0 g}{W^2} - \frac{3\rho_0 u_{0z}}{2W}\right)\right]_{z=0}\right\}. \end{aligned}$$

Note that from (32) $\gamma < 0$ will need $u_{0z}(0) > 0$ and/or $u_{0zz} < 0$ for at least some z.

Here we first consider the surface wave case, when ρ_0 = constant = 1, $N^2 = 0$. Then the solution of (14), (15) is

$$\phi = \int_{-h}^{z}\frac{g}{W^2}dz, \quad \int_{-h}^{0}\frac{g}{W^2}dz = 1, \qquad (33)$$

$$\text{and } \hat{I}\gamma = f^2\left\{\int_{-h}^{0}\frac{3g^2}{2W^4}dz - \frac{g}{2W^2(0)}\right\}, \qquad \hat{I} = \frac{I}{\rho_0} = 2\int_{-h}^{0}\frac{g^2}{W^3}dz. \qquad (34)$$

If $u_0(z) \ge u_0(0)$ then it is readily shown that $\gamma > 0$. But a surface intensified current, that is $u_0(z) \le u_0(0)$ might lead to $\gamma < 0$? Let $u_0 = \alpha(z + h)$ and then (33), (34) yields

$$\frac{g}{\alpha}\left\{\frac{1}{W(0)} - \frac{1}{W(-h)}\right\} = 1, \text{ or } c(c-\alpha h) = gh, \; c = \frac{\alpha h}{2} + \left(gh + \frac{\alpha^2 h^2}{4}\right)^{1/2},$$

$$I\gamma = f^2\left\{\frac{2gh + \alpha^2 h^2 - c\alpha h}{2gh^2}\right\} = f^2\left\{\frac{4gh + \alpha^2 h^2 - \alpha h(gh + \alpha^2 h^2/4)^{1/2}}{4gh^2}\right\},$$

$$I = \frac{2(gh + \alpha^2 h^2/4)^{1/2}}{h}.$$

Hence $\gamma > 0$ for all α, including $\alpha > 0$.

Next, we consider a two-layer fluid, with layer depths $h_{1,2}$ ($h = h_1 + h_2$) and densities $\rho_{1,2}$ with a shear flow $u_0(z)$ in the upper layer, such that $u_0(-h_1) = 0$. Using the Boussinesq approximation with a rigid lid at $z = 0$, the solution of (14), (15) is

$$\phi = \int_z^0 \frac{dz}{W^2}, \qquad 1 = C_1 \int_{-h_1}^0 \frac{dz}{W^2}, \quad -h_1 \le z \le 0,$$

$$\phi = \frac{z+h}{h_2}, \qquad -h \le z \le -h_1,$$

$$C_1 + \frac{c^2}{h_2} = g' = \frac{g(\rho_2 - \rho_1)}{\rho_2},$$

and $$\hat{I}\gamma = f^2 \left\{ \int_{-h_1}^0 \frac{3C_1^2}{2W^4} dz + \frac{1}{h_2} - \frac{C_1}{2c^2} \right\}, \qquad \hat{I} = \frac{I}{\rho_2} = 2\left\{ \int_{-h_1}^0 \frac{C_1^2}{W^3} dz + \frac{c}{h_2} \right\}. \qquad (35)$$

In the limit $h_2 \to \infty$, $C_1 \to g'$ and the expression (35) is quite similar to (34) for surface waves. Hence we conclude again that for a linear shear flow $u_0 = \alpha(z + h_1)$, $\gamma > 0$ for all α.

These two examples indicate that shear flows for which $\gamma < 0$ may be quite hard to find, and a subject for future study.

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